International Journal of Theoretical Physics, Vol. 11, No. 4 (1974), pp. 261-271

Hypercomplex Extensions of the General Linear Group $GL(n, R)$

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Received: 27 *March* 1974

Abstract

Based on the concept of quaternionic extension of general linear group, we consider the case of group extensions using generalised hypercomplex systems such as the Clifford algebra, and the Grassman algebra.

1. *Introduction*

As is well known (Gourdin, 1967; Barut & Baczka, 1965) many interesting classical Lie groups can be obtained as complex or quaternionic extensions of the general real linear group $GL(n,R)$. For example, the groups $GL(n, c)$, $SL(n, C)$, $U(n\text{-}s, s)$, and $O(n, c)$, can be obtained as complex extensions of $GL(n, R)$ or its subgroups, while the groups $GL(n, Q)$, $Sp(n\text{-}s, s)$, $O(n, Q)$ are obtainable as quaternionic extensions of $GL(n, R)$. In general the quaternionic extensions have much richer algebraic and topological structures than the complex extensions, and these in turn are richer than the original real linear groups. By studying the structures of the Lie algebras of these extended groups, one arrives at several useful isomorphisms.

We demonstrate here that all these ideas are amenable to wider generalisations based on the use of generalised hypercomplex systems (Van Der Waerden, 1940; Littlewood, 1958). We end up with a set of new Lie algebras and new Lie groups defined on R^n, the detailed structures of which depend on the exact hypercomplex system used in defining the extensions of $GL(n, R)$.

In order to see the basis for these wider generalisations, we review briefly the case of complex and quaternionic extensions of $GL(n, R)$. Defining the general real linear group $GL(n, R)$ as the set of regular $n \times n$ matrices with real coefficients, its Lie algebra $gl(n, R)$ has the infinitesimal generators $X(x)$ which can be realised as the differential operators:

$$X_{rs} = x_r \frac{\partial}{\partial x^s} \tag{1.1}$$

From this one deduces the defining commutation rules of $gl(n, R)$:

$$[X_{rs}, X_{tu}] = g_{st}X_{ru} - g_{ur}X_{st} \tag{1.2}$$

where g is the symmetrical bilinear connection in R^n. From this one gets such important subalgebras as $sl(n, R)$ with the generators

$$X'_{rs} = X_{rs} - \frac{1}{n} g_{rs}g^{sr}X_{rs}$$

satisfying the same commutation relation (1.2), and the pseudoorthogonal algebra $o(n\text{-}s, s)$, with the commutation relation:

$$[Z_{ij}, Z_{kl}] = g_{jk}Z_{il} - g_{ik}Z_{jl} \tag{1.3}$$

where $Z_{ij} = Z_{ij} - X_{ji} = - Z_{ji}$.

To complexify this algebra $gl(n, R)$, the prescription is to introduce the complex number space with basis elements:

$$(e_0, e_1) \equiv (1, i) \tag{1.4}$$

where $e_0^2 = e_0$; $e_1^2 = -1$; and to replace the set of generators $\{X\}$ of $gl(n, R)$ by the set

$$\{X, Y\}$$

where

$$\left.\begin{aligned} X_{rs} &= e_0 x_r \frac{\partial}{\partial x^s} = x_r \frac{\partial}{\partial x^s} \\ &\text{and} \\ Y_{kl} &= e_1 x_k \frac{\partial}{\partial x^l} = i x_k \frac{\partial}{\partial x^l} \end{aligned}\right\} \tag{1.5}$$

The algebra with the generators $\{X, Y\}$ is the $gl(n, C)$ algebra, the complex extension of $gl(n, R)$. Its commutation relations are easily deducible from (1.5) to be:

$$\begin{aligned} [X_{rs}, X_{tu}] &= g_{st}X_{ru} - g_{ur}X_{ts} \\ [X_{rs}, Y_{tu}] &= g_{st}Y_{ru} - g_{ur}Y_{ts} \\ [Y_{rs}, Y_{tu}] &= - g_{st}X_{ru} + g_{ur}X_{ts} \end{aligned} \tag{1.6}$$

where g is the bilinear connection in C^n. From this we deduce subalgebras like $U(n - s, s)$ of $gl(n, C)$. Also we have the algebra of the complex orthogonal group $O(n, C)$ obtained as the complex extension of $O(n - s, s)$. Thus if the generators of $O(n - s, s)$ are $\{Z_{ij}\}$, the generators of $O(n, C)$ will be the doubled set

$$\{Z_{ij}, Z_{ij}^I\}$$

where

$$Z_{ij} = X_{ij} - X_{ji} \qquad \text{with} \qquad X_{ij} = x_i \frac{\partial}{\partial x^j}$$

and

$$Z^I_{ij} = Y_{ij} - Y_{ji} \qquad \text{with} \qquad Y_{ij} = ix_i \frac{\partial}{\partial x^j}$$

Based on this complexification, we obtain that the Lie algebra of O(n, C) is given by:

$$\begin{aligned}
[Z_{ij}, Z_{kl}] &= g_{jk}Z_{il} - g_{ik}Z_{jl} + g_{il}Z_{jk} - g_{jl}Z_{ik} \\
[Z_{ij}, Z^I_{kl}] &= g_{jk}Z^I_{il} - g_{ik}Z^I_{jl} + g_{il}Z^I_{jk} - g_{jl}Z^I_{ik} \\
[Z^I_{ij}, Z^I_{kl}] &= -g_{jk}Z_{il} + g_{ik}Z_{jl} - g_{il}Z_{jk} + g_{jl}Z_{ik}
\end{aligned} \tag{1.7}$$

2. *Isomorphisms*

Useful isomorphisms can now be established between the original real linear algebras and the complexified algebras. Arguing that a necessary (though not sufficient) condition for isomorphism to exist between two algebras is that they must have the same number of generators, one finds the following well-known isomorphisms:

$$\begin{aligned}
&su(2) \simeq so(3); \\
&su(1, 1) \simeq so(2, 1) \simeq sl(2, R) \\
&so(3, C) \simeq so(3, 1) \simeq sl(2, C) \\
&so(4, C) \simeq so(3, 1) \otimes so(3, 1) \\
&su(2, 2) \simeq so(4, 2) \\
&sl(4, R) \simeq so(3, 3)
\end{aligned} \tag{2.1}$$

For example, to establish isomorphism between $sl(n, R)$ and $so(m - s, s)$ we first find a set of positive integers n and m which will satisfy the following relation:

$$n^2 - 1 = \frac{m}{2}(m - 1)$$

that is

$$m = \frac{1 + \sqrt{(8n^2 - 7)}}{2}. \tag{2.2}$$

Some of the solutions are:

(i) $n = 2, m = 3$, leading to the isomorphism: $sl(2, R) \simeq so(3 - s, s)$ for some s. The exact value of s has to be determined by looking at the commutation relations of the two Lie algebras. One finds $sl(2, R) \simeq so(2, 1)$.

(ii) Similarly, we have a solution for $n = 4$ and $m = 6$, leading to $sl(4, R) \simeq so(6 - s, s)$, for some s. Again we find that $s = 3$, so that $sl(4, R) \simeq so(3, 3)$.

By setting up such constraint equations between the generators of any two Lie algebras between which we are seeking isomorphism, we can infer all the isomorphisms listed above.

3. *Quaternionic Extensions*

We consider next the quaternionic extensions of $gl(n, R)$. As is well known, just as a complex number is defined in terms of two basic units $(e_0, e_1) \equiv (1, i)$, so also a quaternion is a number defined in terms of four basic units: e_0, e_1, e_2 and e_3, so that

$$q = a_0 e_0 + \sum_{j=1}^{3} a_j e_j, \qquad j = 1, 2, 3 \tag{3.1}$$

Like the complexification unit i, satisfying $i^2 = -1$, we have $e_j^2 = -1$, so that the conjugate quaternion is to be defined as

$$q^* = a_0 e_0 - \sum_{j=1}^{3} a_j e_j$$

The complete multiplication table for these quaternion units is shown in Table 1

Now in general if $\{X\}$ are the generators of a given Lie algebra, its quaternionic extension is defined as the Lie algebra with the generators

$$\{X, Y_{e_1}, Y_{e_2}, Y_{e_3}\} \tag{3.2}$$

where for $X = X_{rs} = x_r(\partial/\partial x^s)$

$$Y_{e_j} = (Y_{ej})_{rs} = e_j x_r \frac{\partial}{\partial x^s}, \qquad j = 1, 2, 3$$

The commutation relations of the new Lie algebra $gl(n, Q)$ can now be obtained by using these differential forms for the generators plus the above multiplication

TABLE 1

X	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	$-e_0$	e_3	$-e_2$
e_2	e_2	$-e_3$	$-e_0$	e_1
e_3	e_3	e_2	$-e_1$	$-e_0$

Also $e_i e_j = -e_j e_i$, $i, j = 1, 2, 3$, $i \neq j$.

table. We find that the quaternionic extension of $gl(n, R)$ is the algebra with the commutation rules:

$$
\begin{aligned}
&[X_{jk}, X_{lm}] = g_{kl}X_{jm} - g_{mj}X_{lk} \\
&[X_{jk}, Y^{(\alpha)}_{lm}] = g_{kl}Y^{(\alpha)}_{jm} - g_{mj}Y^{(\alpha)}_{lk} \qquad \alpha = 1, 2, 3 \\
&[Y^{(\alpha)}_{jk}, Y^{(\alpha)}_{lm}] = -g_{kl}X_{jm} + g_{mj}X_{lk}, \qquad \text{for a fixed } \alpha \\
&[Y^{(\alpha)}_{jk}, Y^{(\beta)}_{lm}] = -\epsilon_{\alpha\beta\gamma}(g_{kl}Y^{(\gamma)}_{jm} + g_{mj}Y^{(\gamma)}_{lk}), \qquad \alpha \neq \beta
\end{aligned} \tag{3.3}
$$

From this one deduces the commutation relations of quaternionic subalgebras like the pseudo-symplectic algebra $sp(n - s, s)$ and the orthogonal quaternionic algebra $o(n, Q)$. Finally we note that one can complexify the $gl(n, Q)$ algebra by using the set of generators:

$$\{X, X^I, Y_{e_1}, Y^I_{e_1}, Y_{e_2}, Y^I_{e_2}, Y_{e_3}, Y^I_{e_3}\}$$

where

$$(X^I)_{rs} = ix_r \frac{\partial}{\partial x^s}$$

$$(Y^I_{e_j})_{rs} = ie_j x_r \frac{\partial}{\partial x^s}, \qquad j = 1, 2, 3$$

where i is the complexification unit and e_j are quaternion units.

4. *Isomorphisms*

Again by setting up elementary constraint relations between the numbers of generators of any two algebras, we obtain a host of new isomorphisms such as:

$$
\begin{aligned}
&sl(2, Q) \simeq so(5, 1);\ sp(2) \simeq so(5) \\
&so(3, Q) \simeq su(3, 1);\ sp(1, 1) \simeq so(4, 1) \\
&so(4, Q) \simeq so(6, 2);\ sp(1) \simeq su(2) \simeq so(3) \\
&\qquad sp(1, C) \simeq so(3, 1) \simeq sl(2, C) \\
&sp(1, R) \simeq sl(2, R) \simeq so(2, 1) \simeq su(1, 1)
\end{aligned}
$$

5. *Pauli Extension*

We note that the quaternionic extension of a given Lie algebra as defined above, can also be called the Pauli extension of that algebra. This follows from the fact that the quaternionic units are in 1-1 correspondence with the Pauli units as follows:

$$e_0 \leftrightarrow \sigma_0; \qquad e_j \leftrightarrow i\sigma_j, \qquad j = 1, 2, 3$$

This leads to the familiar two-dimensional representation of a quaternion as :

$$q = (x, y) = \begin{pmatrix} x, & \bar{y} \\ -y, & \bar{x} \end{pmatrix}$$

where x and y are complex numbers. The algebra $gl(n, Q)$ may therefore be denoted also by $gl(n, P)$.

6. *Generalisation to Arbitrary Hypercomplex Systems*

Having now reviewed these familiar Lie algebraic extensions, it becomes obvious that by using higher hypercomplex systems, one can generate a host of new Lie algebraic structures, defined on R^n. All that is required, is to use a hypercomplex system with a base unit e_0 among its set of bases, where e_0 has to behave like the real number unit. Consider now one such hypercomplex system (H) with bases denoted by

$$e_0 = 1 \qquad \text{and} \qquad e_j, \qquad j = 1, 2, \ldots, n$$

We define the H-extension of a given Lie algebra $\{X\}$ as the algebra $gl(n, H)$ with the generators

$$X, Y_{e_1}, Y_{e_2} \ldots, Y_{e_n}$$

where

$$X = X_{rs} = x_r \frac{\partial}{\partial x^s}$$

and

$$Y_{e_j} = e_j x_r \frac{\partial}{\partial x^s}$$

If we know the multiplication table of the H system, we can deduce the explicit commutation relations of the new Lie algebra.

The associated Lie group may be denoted by $GL(n, H)$ and may be considered as the set of $n \times n$ matrices in which each coefficient is a hypercomplex number h given by

$$h = \sum_{j=0}^{n} a_j e_j$$

The properties of such n-dim. hypercomplex spaces H^n are themselves interesting, and lead us to consider generalised symplectic geometry.

Now by studying the structure of the algebras $gl(n, H)$ we can deduce various Lie isomorphisms, and how these results depend on the hypercomplex system H used. Since one can immediately think of several hypercomplex systems—ranging from a host of Clifford algebras, generalised Clifford algebras, Grassman algebras, etc, the problem can easily become complicated. We illustrate however the new features by considering a tractable case— the case of Dirac extensions of $gl(n, R)$.

7. *Dirac Hypercomplex System*

We consider Lie algebraic extensions based on the Dirac hypercomplex system. This system has sixteen elements:

$$1, \gamma_\mu, \gamma_5, \sigma_{\mu\nu} = \frac{1}{2i}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \qquad \text{and} \qquad \gamma_\mu\gamma_5$$

We shall choose the sixteen units as

$$\begin{aligned}
\Gamma_0 &= 1\\
\Gamma_\mu &= \gamma_\mu, \qquad \mu = 1, 2, 3, 4, 5\\
\Gamma_6 &= -\gamma_2\gamma_3\gamma_4\\
\Gamma_7 &= \gamma_1\gamma_3\gamma_4\\
\Gamma_8 &= -\gamma_1\gamma_2\gamma_4\\
\Gamma_9 &= \gamma_1\gamma_2\gamma_3\\
\Gamma_{10} &= -i\gamma_1\gamma_2\\
\Gamma_{11} &= -i\gamma_1\gamma_3\\
\Gamma_{12} &= -i\gamma_1\gamma_4\\
\Gamma_{13} &= -i\gamma_2\gamma_3\\
\Gamma_{14} &= -i\gamma_2\gamma_4\\
\Gamma_{15} &= -i\gamma_3\gamma_4
\end{aligned}$$

The multiplication table of these Dirac units is as shown in Table 2.

Analogous to the quaternionic number

$$q = (a_0, \mathbf{a}) = \sum_{j=0}^{3} a_j e_j$$

one introduces the Dirac hypercomplex number

$$D = (a_0, \mathbf{a}) = \sum_{j=0}^{15} a_j \Gamma_j$$

Also corresponding to the n-dimensional quaternionic space Q^n which is equivalent to a $4n$-dimensional real vector space R^{4n}, we can think of an n-dimentional Dirac space D^n equivalent to a $16n$-dimensional real vector space R^{16n}. In such a space we can consider the norms and scalar products of

TABLE 2

$i\downarrow \rightarrow j : (i \times j)$

X	Γ_0	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9	Γ_{10}	Γ_{11}	Γ_{12}	Γ_{13}	Γ_{14}	Γ_{15}
Γ_0	Γ_0	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9	Γ_{10}	Γ_{11}	Γ_{12}	Γ_{13}	Γ_{14}	Γ_{15}
Γ_1	Γ_1	Γ_0	$i\Gamma_{10}$	$i\Gamma_{11}$	$i\Gamma_{12}$	$-\Gamma_6$	$-\Gamma_5$	$i\Gamma_{15}$	$-i\Gamma_{14}$	$i\Gamma_{13}$	$-i\Gamma_2$	$-i\Gamma_3$	$-i\Gamma_4$	$-i\Gamma_9$	$i\Gamma_8$	$-i\Gamma_7$
Γ_2	Γ_2	$-i\Gamma_{10}$	Γ_0	$i\Gamma_{13}$	$i\Gamma_{14}$	$-\Gamma_7$	$-i\Gamma_{15}$	$-\Gamma_5$	$i\Gamma_{12}$	$-i\Gamma_{11}$	$i\Gamma_1$	$i\Gamma_9$	$-i\Gamma_8$	$-i\Gamma_3$	$-i\Gamma_4$	$i\Gamma_6$
Γ_3	Γ_3	$-i\Gamma_{11}$	$-i\Gamma_{13}$	Γ_0	$i\Gamma_{15}$	$-\Gamma_8$	$i\Gamma_{14}$	$-i\Gamma_{12}$	$-\Gamma_5$	$i\Gamma_{10}$	$-i\Gamma_9$	$i\Gamma_1$	$i\Gamma_7$	$i\Gamma_2$	$-i\Gamma_6$	$-i\Gamma_4$
Γ_4	Γ_4	$-i\Gamma_{12}$	$-i\Gamma_{14}$	$-i\Gamma_{15}$	Γ_0	$-\Gamma_9$	$-i\Gamma_{13}$	$i\Gamma_{11}$	$-i\Gamma_{10}$	$-\Gamma_5$	$i\Gamma_8$	$-i\Gamma_7$	$i\Gamma_1$	$i\Gamma_6$	$i\Gamma_2$	$i\Gamma_3$
Γ_5	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9	Γ_0	Γ_1	Γ_2	Γ_3	Γ_4	$-\Gamma_{15}$	Γ_{14}	$-\Gamma_{13}$	$-\Gamma_{12}$	Γ_{11}	$-\Gamma_{10}$
Γ_6	Γ_6	Γ_5	$-i\Gamma_{15}$	$i\Gamma_{14}$	$-i\Gamma_{13}$	$-\Gamma_1$	$-\Gamma_0$	$-i\Gamma_{10}$	$-i\Gamma_{11}$	$-i\Gamma_{12}$	$-i\Gamma_7$	$-i\Gamma_8$	$-i\Gamma_9$	$-i\Gamma_4$	$i\Gamma_3$	$-i\Gamma_2$
Γ_7	Γ_7	$i\Gamma_{15}$	Γ_5	$-i\Gamma_{12}$	$i\Gamma_{11}$	$-\Gamma_2$	$i\Gamma_{10}$	$-\Gamma_0$	$-i\Gamma_{13}$	$-i\Gamma_{14}$	$i\Gamma_6$	$i\Gamma_4$	$-i\Gamma_3$	$-i\Gamma_8$	$-i\Gamma_9$	$i\Gamma_1$
Γ_8	Γ_8	$-i\Gamma_{14}$	$i\Gamma_{12}$	Γ_5	$-i\Gamma_{10}$	$-\Gamma_3$	$i\Gamma_{11}$	$i\Gamma_{13}$	$-\Gamma_0$	$-i\Gamma_{15}$	$-i\Gamma_4$	$i\Gamma_6$	$i\Gamma_2$	$i\Gamma_7$	$-i\Gamma_1$	$-i\Gamma_9$
Γ_9	Γ_9	$i\Gamma_{13}$	$-i\Gamma_{11}$	$i\Gamma_{10}$	Γ_5	$-\Gamma_4$	$i\Gamma_{12}$	$i\Gamma_{14}$	$i\Gamma_{15}$	$-\Gamma_0$	$i\Gamma_3$	$-i\Gamma_2$	$i\Gamma_6$	$i\Gamma_1$	$i\Gamma_7$	$i\Gamma_8$
Γ_{10}	Γ_{10}	$i\Gamma_2$	$-i\Gamma_1$	$-i\Gamma_9$	Γ_{14}	$-\Gamma_{15}$	$i\Gamma_7$	$-i\Gamma_6$	$-i\Gamma_4$	$i\Gamma_3$	Γ_0	$i\Gamma_{13}$	$i\Gamma_{14}$	$-i\Gamma_{11}$	$-i\Gamma_{12}$	$-\Gamma_5$
Γ_{11}	Γ_{11}	$i\Gamma_3$	$i\Gamma_9$	$-i\Gamma_1$	$-i\Gamma_7$	Γ_{14}	$i\Gamma_8$	$i\Gamma_4$	$-i\Gamma_6$	$-i\Gamma_2$	$-i\Gamma_{13}$	Γ_0	$i\Gamma_{15}$	$i\Gamma_{10}$	Γ_5	$-i\Gamma_{12}$
Γ_{12}	Γ_{12}	$i\Gamma_4$	$-i\Gamma_8$	$i\Gamma_7$	$-i\Gamma_1$	$-\Gamma_{13}$	$i\Gamma_9$	$-i\Gamma_3$	$i\Gamma_2$	$-i\Gamma_6$	$-i\Gamma_{14}$	$-i\Gamma_{15}$	Γ_0	$-\Gamma_5$	$i\Gamma_{10}$	$i\Gamma_{11}$
Γ_{13}	Γ_{13}	$-i\Gamma_9$	$i\Gamma_3$	$-i\Gamma_2$	$i\Gamma_6$	$-\Gamma_{12}$	$-i\Gamma_4$	$i\Gamma_8$	$-i\Gamma_7$	$i\Gamma_1$	$i\Gamma_{11}$	$-i\Gamma_{10}$	$-\Gamma_5$	Γ_0	$i\Gamma_{15}$	$-i\Gamma_{14}$
Γ_{14}	Γ_{14}	$i\Gamma_8$	$i\Gamma_4$	$-i\Gamma_6$	$-i\Gamma_2$	Γ_{11}	$i\Gamma_3$	$i\Gamma_9$	$-i\Gamma_1$	$-i\Gamma_7$	$i\Gamma_{12}$	Γ_5	$-i\Gamma_{10}$	$-i\Gamma_{15}$	Γ_0	$i\Gamma_{13}$
Γ_{15}	Γ_{15}	$-i\Gamma_7$	$i\Gamma_6$	$i\Gamma_4$	$-i\Gamma_3$	$-\Gamma_{10}$	$-i\Gamma_2$	$i\Gamma_1$	$i\Gamma_9$	$-i\Gamma_8$	$-\Gamma_5$	$i\Gamma_{12}$	$-i\Gamma_{11}$	$i\Gamma_{14}$	$-i\Gamma_{13}$	Γ_0

two vectors. Thus if u and v are two vectors in D^n, we can introduce a metric connection g_{ik} in D^n such that

$$(u, v) = u^{*i} g_{ik} v^k$$

To get the norm of a vector in D^n we need to define the conjugate number D^* given by:

$$D^* = (a_0; a_1 \ldots, a_5; -a_6, -a_7, -a_8, -a_9; a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15})$$

so that

$$DD^* = \sum_{j=0}^{5} a_j^2 + \sum_{j=6}^{9} a_j^2 + \sum_{j=10}^{15} a_j^2$$

We note that the above form of D^* is determined by the number of the basis elements of the Dirac hypercomplex system with squares equal to ±1. From the table we see that $\Gamma_0{}^2$ to $\Gamma_5{}^2$ give +1; $\Gamma_6{}^2$ to $\Gamma_9{}^2$ give −1, while $\Gamma_{10}{}^2$ to $\Gamma_{15}{}^2$ give +1. From these one then constructs complex conjugate of the Dirac hypercomplex number D as given above.

Similarly, we can construct the product of two Dirac numbers. The familiar case of two quaternions is as follows: If

$$q_1 = \sum_{j=0}^{3} a_j e_j$$

$$q_2 = \sum_{k=0}^{3} b_k e_k$$

then

$$q_1 q_2 = \left(a_0 e_0 + \sum_{j=1}^{3} a_j e_j\right)\left(b_0 e_0 + \sum_{k=1}^{3} b_k e_k\right)$$
$$= (a_0 b_0 - \mathbf{a}\,.\,\mathbf{b};\ b_0 \mathbf{a} + a_0 \mathbf{b} + \mathbf{a}_\wedge \mathbf{b})$$

where we have made use of the multiplication table for the quaternionic units. In the same way the product of two Dirac hypercomplex numbers D_1 and D_2 can be written as

$$D_1 D_2 = \left(\sum_{j=0}^{15} a_j \Gamma_j\right)\left(\sum_{k=0}^{15} b_k \Gamma_k\right)$$
$$= \sum_{j=0}^{15} \sum_{k=0}^{15} a_j b_k \Gamma_j \Gamma_k$$

where the product $\Gamma_j \Gamma_k$ can now be evaluated from the multiplication table of the Dirac units.

Finally, corresponding to the 2 x 2 metric representation of a quaternion, one writes the 4 x 4 matrix representation of a Dirac hypercomplex number.

8. *The Lie Algebra gl(n, D)*

We now consider the Dirac extended algebra $gl(n, D)$. The corresponding group is $GL(n, D)$, the set of $n \times n$ matrices with D coefficients. If we denote the generators of $GL(n, R)$ by $\{X\}$ where $X = X_{rs} = x_r\,(\partial/\partial x^s)$, then the generators of $Gl(n, D)$ will be made up of the following sixteen sets of generators X and $Y^{(\alpha)}$, $\alpha = 1, 2, 3, \ldots, 15$, where

$$X = X_{rs} = x_r \frac{\partial}{\partial x^s}$$

and

$$Y^{(\alpha)} = (Y^{(\alpha)})_{rs} = \Gamma_\alpha x_r \frac{\partial}{\partial x^s}$$

Using this form and the multiplication table for the Γ_α, we deduce the following commutation relations for the Lie algebra $gl(n, D)$.

$$[X_{rs}, X_{kl}] = g_{sk} X_{rl} - g_{lr} X_{ks}$$

$$[X_{rs}, Y^\alpha_{kl}] = g_{sk} Y^\alpha_{rl} - g_{lr} Y^\alpha_{ks}$$

$$[Y^\alpha_{rs}, Y^\alpha_{lk}] = \epsilon(\alpha)(g_{sk} X_{rl} - g_{lr} X_{ks}),$$

where $\epsilon(\alpha) = +1$, $\alpha = 1, 2, 3, 4, 5, 10, 11, 12, 13, 14, 15$

and $\epsilon(\alpha) = -1$, $\alpha = 6, 7, 8, 9$

$$[Y^\alpha_{rs}, Y^\beta_{kl}] = \Gamma_\alpha \Gamma_\beta x_r g_{sk} \frac{\partial}{\partial x^l} - \Gamma_\beta \Gamma_\alpha x_k g_{lr} \frac{\partial}{\partial x^s} \quad (\alpha \neq \beta).$$

Examples:

$$[Y^1_{rs}, Y^2_{lk}] = i(g_{sk} Y^{10}_{lr} + g_{lr} Y^{10}_{ks})$$

$$[Y^1_{rs}, Y^3_{kl}] = i(g_{sk} Y^{11}_{rl} + g_{lr} Y^{11}_{ks})$$

$$[Y^1_{rs}, Y^8_{kl}] = i(g_{sk} Y^{14}_{rl} - g_{lr} Y^{14}_{ks})$$

$$[Y^1_{rs}, Y^{15}_{kl}] = -i(g_{sk} Y^7_{rl} - g_{lr} Y^7_{ks})$$

etc.

9. *Isomorphisms*

Using the principles discussed earlier, one can now look for isomorphisms between these new Lie algebraic structures and the familiar classical Lie algebras. For example, since the algebra $gl(n, D)$ has $(16n^2 - 1)$ generators, solving the equation:

$$16n^2 - 1 = \frac{m}{2}(m - 1)$$

we get one solution: $n = 1$, $m = 6$, so that we expect isomorphism between $Sl(1, D)$ and $So(6 - s, s)$. As before, one finds the correct value of the signature s by looking at the actual commutation relations of the two Lie algebras. Other isomorphisms can be established in the same way.

10. *Conclusion*

We now see that given a suitable hypercomplex system one can always define generalised extensions of real Lie algebras. In fact one can generate an infinite set of such algebras. The topological properties of the associated Lie groups $GL(n, H)$ and the hypercomplex geometry of the H^n spaces, are of interest and will be discussed in a subsequent paper.

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